Ellipse

1. Definitions

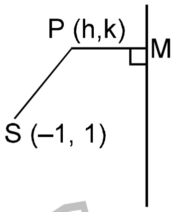
It is a locus of a point which moves in such a way that the ratio of its distance from a fixed point and a fixed line (not passes through fixed point and all points and line lies in same plane) is constant (e) which is less than one.
The fixed point is called - **focus**
The fixed line is called -**directrix.**
The constant ratio is called - **eccentricity**, it is denoted by **'e'**.

Solved Example # 1

Find the equation to the ellipse whose focus is the point (– 1, 1), whose directrix is the straight line x – y + 3 = 0 and eccentricity is $\frac{1}{2}$.

Solution.

Let P ≡ (h, k) be moving point,



$$e = \frac{PS}{PM} = \frac{1}{2}$$

$$\Rightarrow \quad (h + 1)^2 + (k - 1)^2 = \frac{1}{4}\left(\frac{h-k+3}{\sqrt{2}}\right)^2$$

⇒ locus of P(h, k) is

$$8\{x^2 + y^2 + 2x - 2y + 2\} = (x^2 + y^2 - 2xy + 6x - 6y + 9)$$

$$7x^2 + 7y^2 + 2xy + 10x - 10y + 7 = 0.$$ **Ans.**

Note : The general equation of a conic with focus (p, q) & directrix lx + my + n = 0 is:

$$(l^2 + m^2)[(x - p)^2 + (y - q)^2] = e^2(lx + my + n)^2$$

$$\equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

represent ellipse if $0 < e < 1$; $\Delta \neq 0$, $h^2 < ab$

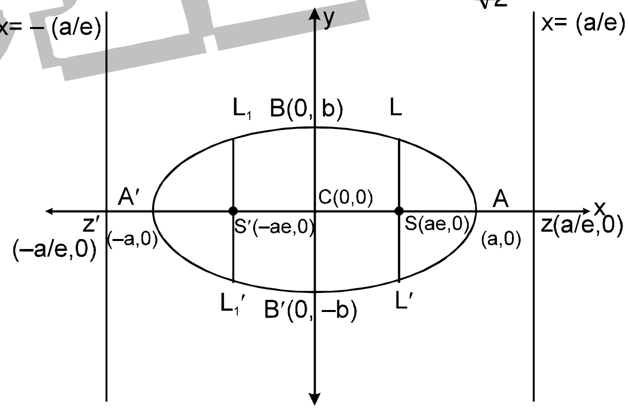
Self Practice Problem :-

1. Find the equation to the ellipse whose focus is (0, 0) directrix is x + y – 1 = 0 and $e = \frac{1}{\sqrt{2}}$.

Ans. $3x^2 + 3y^2 - 2xy + 2x + 2y - 1 = 0$.

2. Standard Equation

Standard equation of an ellipse referred to its principal axes along the co–ordinate axes is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where a > b & $b^2 = a^2(1 - e^2)$.



Eccentricity: $e = \sqrt{1 - \frac{b^2}{a^2}}$, $(0 < e < 1)$

Focii : S ≡ (a e, 0) & S′ ≡ (– a e, 0).

Equations of Directrices : $x = \frac{a}{e}$ & $x = -\frac{a}{e}$.

Major Axis : The line segment A′A in which the focii S′ & S lie is of length 2a & is called the major axis (a > b) of the ellipse. Point of intersection of major axis with directrix is called the foot of the directrix (Z).

Minor Axis : The y–axis intersects the ellipse in the points B′ ≡ (0, – b) & B ≡ (0, b). The line segment B′B is of length 2b (b < a) is called the minor axis of the ellipse.

Principal Axis : The major & minor axes together are called principal axis of the ellipse.

Vertices : Point of intersection of ellipse with major axis. A′ ≡ (– a, 0) & A ≡ (a, 0) .

Focal Chord : A chord which passes through a focus is called a focal chord.

Double Ordinate : A chord perpendicular to the major axis is called a double ordinate.

Latus Rectum : The focal chord perpendicular to the major axis is called the latus rectum.

Length of latus rectum **(LL′)** = $\frac{2b^2}{a} = \frac{(\text{minor axis})^2}{\text{major axis}} = 2a(1 - e^2)$

= 2 e (distance from focus to the corresponding directrix)

Centre : The point which bisects every chord of the conic drawn through it, is called the centre of the conic. C ≡ (0, 0) the origin is the centre of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

NOTE :

(i) If the equation of the ellipse is given as $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and nothing is mentioned, then the rule is to assume that a > b.

(ii) If b > a is given, then the y–axis will become major axis and x-axis will become the minor axis and all other points and lines will change accordingly.

Solved Example # 2: Find the equation to the ellipse whose centre is origin, axes are the axes of co-ordinate and passes through the points (2, 2) and (3, 1).

Solution. Let the equation to the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Since it passes through the points (2, 2) and (3, 1)

$$\therefore \quad \frac{4}{a^2} + \frac{4}{b^2} = 1 \quad ....(i)$$

and $$\frac{9}{a^2} + \frac{1}{b^2} = 1 \quad ....(ii)$$

from (i) – 4 (ii), we get

$$\frac{4-36}{a^2} = 1 - 4 \Rightarrow \quad a^2 = \frac{32}{3}$$

from (i), we get

$$\frac{1}{b^2} = \frac{1}{4} - \frac{3}{32} = \frac{8-3}{32}$$

$$b^2 = \frac{32}{5} \qquad \therefore \qquad \text{Ellipse is } 3x^2 + 5y^2 = 32$$ **Ans.**

Solved Example # 3

Find the equation of the ellipse whose focii are (4, 0) and (– 4, 0) and eccentricity is $\frac{1}{3}$

Solution. Since both focus lies on x-axis, therefore x-axis is major axis and mid point of focii is origin which is centre and a line perpendicular to major axis and passes throguh centre is minor axis which is y-axis.
Let equation of ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$\because$ ae = 4 and $e = \frac{1}{3}$ (Given)

$\therefore$ a = 12 and $b^2 = a^2 (1 - e^2)$

$\Rightarrow$ $b^2 = 144 \left(1 - \frac{1}{9}\right)$

$b^2 = 16 \times 8$

$b = 8\sqrt{2}$

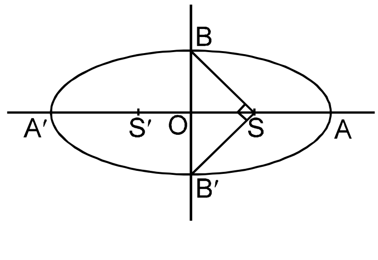
Equation of ellipse is $\frac{x^2}{144} + \frac{y^2}{128} = 1$ **Ans.**

Solved Example # 4

If minor-axis of ellipse subtend a right angle at its focus then find the eccentricity of ellipse.

Solution.

Let the equation of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (a > b)



$\because$ $\angle BSB' = \frac{\pi}{2}$

and OB = OB′

$\therefore$ $\angle BSO = \frac{\pi}{4}$

$\Rightarrow$ OS = OB $\Rightarrow$ ae = b

$\Rightarrow \quad e^2 = \frac{b^2}{a^2} = 1 - e^2$

$\Rightarrow \quad e = \frac{1}{\sqrt{2}}$ **Ans.**

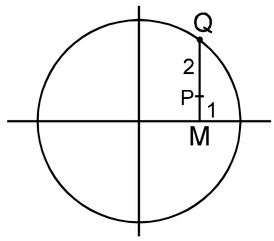
Solved Example # 5: From a point Q on the circle $x^2 + y^2 = a^2$, perpendicular QM are drawn to x-axis, find the locus of point 'P' dividing QM in ratio 2 : 1.

Solution.

Let $Q \equiv (a \cos\theta, a \sin\theta)$
$M \equiv (a \cos\theta, 0)$

Let $P \equiv (h, k)$

$\therefore \quad h = a \cos\theta, k = \frac{a \sin\theta}{3}$



$\therefore \quad \left(\frac{3k}{a}\right)^2 + \left(\frac{h}{a}\right)^2 = 1$

$\Rightarrow$ Locus of P is $\frac{x^2}{a^2} + \frac{y^2}{(a/3)^2} = 1$ **Ans.**

Solved Example # 6

Find the equation of axes, directrix, co-ordinate of focii, centre, vertices, length of latus - rectum and eccentricity of an ellipse $\frac{(x-3)^2}{25} + \frac{(y-2)^2}{16} = 1$.

Solution. Let $x - 3 = X$, $y - 2 = Y$, so equation of ellipse becomes as $\frac{X^2}{5^2} + \frac{Y^2}{4^2} = 1$.

equation of major axis is Y = 0 $\Rightarrow$ y = 2.
equation of minor axis is X = 0 $\Rightarrow$ x = 3.
centre (X = 0, Y = 0) $\Rightarrow$ x = 3, y = 2
C ≡ (3, 2)

Length of semi-major axis a = 5
Length of major axis 2a = 10
Length of semi-minor axis b = 4
Length of major axis = 2b = 8.
Let 'e' be eccentricity

$\therefore \quad b^2 = a^2 (1 - e^2)$

$e = \sqrt{\frac{a^2 - b^2}{a^2}} = \sqrt{\frac{25-16}{25}} = \frac{3}{5}$.

Length of latus rectum $= LL' = \frac{2b^2}{a} = \frac{2\times16}{5} = \frac{32}{5}$

Co-ordinates focii are $X = \pm ae$, $Y = 0$

$\Rightarrow \quad S \equiv (X = 3, Y = 0) \quad \& \quad S' \equiv (X = -3, Y = 0)$

$\Rightarrow \quad S \equiv (6, 2) \quad \& \quad S' \equiv (0, 2)$

<u>Co-ordinate of vertices</u>

Extremities of major axis $A \equiv (X = a, Y = 0)$ & $A' \equiv (X = -a, Y = 0)$
$\Rightarrow \quad A \equiv (x = 8, y = 2)$ & $A' = (x = -2, 2)$
$A \equiv (8, 2)$ & $A' \equiv (-2, 2)$
Extremities of minor axis $B \equiv (X = 0, Y = b)$ & $B' \equiv (X = 0, Y = -b)$
$B \equiv (x = 3, y = 6)$ & $B' \equiv (x = 3, y = -2)$
$B \equiv (3, 6)$ & $B' \equiv (3, -2)$

Equation of directrix $X = \pm \frac{a}{e}$

$x - 3 = \pm \frac{25}{3} \quad \Rightarrow \quad x = \frac{34}{3} \quad \& \quad x = -\frac{16}{3}$

Self Practice Problem

2. Find the equation to the ellipse whose axes are of lengths 6 and $2\sqrt{6}$ and their equations are $x - 3y + 3 = 0$ and $3x + y - 1 = 0$ respectively.

Ans. $3(x - 3y + 3)^2 + 2(3x + y - 1)^2 = 180$, $\quad 21x^2 - 6xy + 29y^2 + 6x - 58y - 151 = 0$.

3. Find the eccentricity of ellipse whose minor axis is double the latus rectum.

Ans. $\frac{\sqrt{3}}{2}$

4. Find the co-ordinates of the focii of the ellipse $4x^2 + 9y^2 = 1$.

Ans. $\left(\pm\frac{\sqrt{5}}{6}, 0\right)$

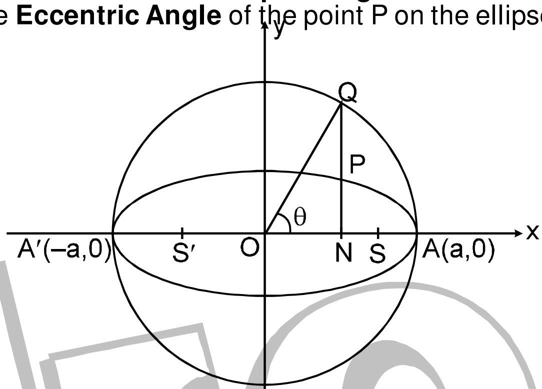
5. Find the standard ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ passing through (2, 1) and having eccentricity $\frac{1}{2}$.

Ans. $3x^2 + 4y^2 = 16$

6. A point moves so that the sum of the squares of its distances from two intersecting non perpendicular straight lines is constant. Prove that its locus is an ellipse.

3. Auxiliary Circle / Eccentric Angle :

A circle described on major axis of ellipse as diameter is called the **auxiliary circle**.
Let Q be a point on the auxiliary circle $x^2 + y^2 = a^2$ such that line through Q perpendicular to the x – axis on the way intersects the ellipse at P, then P & Q are called as the **Corresponding Points** on the ellipse & the auxiliary circle respectively. 'θ' is called the **Eccentric Angle** of the point P on the ellipse $(-\pi < \theta \le \pi)$. $Q \equiv (a\cos\theta, a\sin\theta)$
$P \equiv (a\cos\theta, b\sin\theta)$



Note that :

$$\frac{\ell(PN)}{\ell(QN)} = \frac{b}{a} = \frac{\text{Semi minor axis}}{\text{Semi major axis}}$$

NOTE : If from each point of a circle perpendiculars are drawn upon a fixed diameter then the locus of the points dividing these perpendiculars in a given ratio is an ellipse of which the given circle is the auxiliary circle.

Solved Example # 7

Find the focal distance of a point P(θ) on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (a > b)

Solution.
Let 'e' be the eccentricity of ellipse.

$\therefore$ PS = e . PM

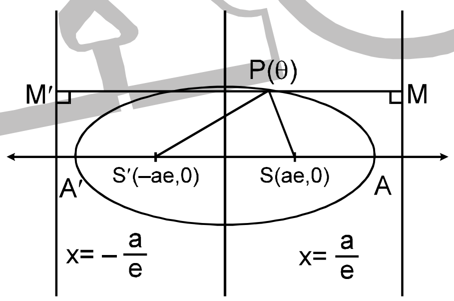
$= e\left(\frac{a}{e} - a\cos\theta\right)$

PS = (a – a e cosθ)

and PS′ = e. PM′

$= e\left(a\cos\theta + \frac{a}{e}\right)$

PS′ = a + ae cosθ



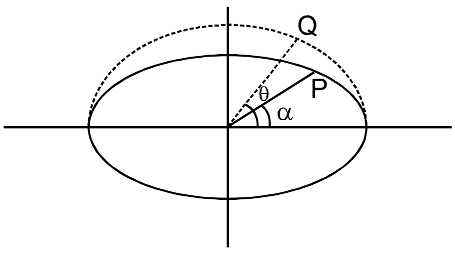
$\therefore$ focal distance are (a ± ae cosθ)

Note : PS + PS′ = 2a
PS + PS′ = AA′

Solvex Eample # 8

Find the distance from centre of the point P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ whose radius makes angle α with x-axis.

Sol. Let $P \equiv (a\cos\theta, b\sin\theta)$



$\therefore \frac{b}{a}\tan\theta = \tan\alpha$

$\tan\theta = \frac{a}{b}\tan\alpha$

$$OP = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta} = \sqrt{\frac{a^2 + b^2\tan^2\theta}{\sec^2\theta}}$$

$$= \sqrt{\frac{a^2 + b^2\tan^2\theta}{1 + \tan^2\theta}} = \sqrt{\frac{a^2 + b^2 \times \frac{a^2}{b^2}\tan^2\alpha}{1 + \frac{a^2}{b^2}\tan^2\alpha}}$$

$$OP = \frac{ab}{\sqrt{a^2\sin^2\alpha + b^2\cos^2\alpha}}$$ **Ans.**

Self Practice Problem

7. Find the distance from centre of the point P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ whose eccentine angle is α

Ans. $r = \sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha}$

8. Find the eccentric angle of a point on the ellipse $\frac{x^2}{6} + \frac{y^2}{2} = 1$ whose distance from the centre is 2.

Ans. $\pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$

9. Show that the area of triangle inscribed in an ellipse bears a constant ratio to the area of the triangle formed by joining points on the auxiliary circle corresponding to the vertices of the first triangle.

4. Parametric Representation:

The equations $x = a\cos\theta$ & $y = b\sin\theta$ together represent the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Where θ is a parameter. Note that if $P(\theta) \equiv (a\cos\theta, b\sin\theta)$ is on the ellipse then; $Q(\theta) \equiv (a\cos\theta, a\sin\theta)$ is on the auxiliary circle.

The equation to the chord of the ellipse joining two points with eccentric angles α & β is given by $\frac{x}{a}\cos\frac{\alpha+\beta}{2} + \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha-\beta}{2}$.

Solved Example # 9

Write the equation of chord of an ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ joining two points P $\left(\frac{\pi}{4}\right)$ and Q $\left(\frac{5\pi}{4}\right)$.

Solution.

Equation of chord is

$$\frac{x}{5}\cos\frac{\left(\frac{\pi}{4}+\frac{5\pi}{4}\right)}{2} + \frac{y}{4}\cdot\sin\frac{\left(\frac{\pi}{4}+\frac{5\pi}{4}\right)}{2} = \cos\frac{\left(\frac{\pi}{4}-\frac{5\pi}{4}\right)}{2}$$

$$\frac{x}{5}\cdot\cos\left(\frac{3\pi}{4}\right) + \frac{y}{4}\cdot\sin\left(\frac{3\pi}{4}\right) = 0$$

$$-\frac{x}{5} + \frac{y}{5} = 0 \quad \Rightarrow \quad y = x$$ **Ans.**

If $P(\alpha)$ and $P(\beta)$ are extremities of a focal chord of ellipse then prove that its eccentricity

$$e = \left|\frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)}\right|.$$

Solution.

Let the equation of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\therefore$ equation of chord is

$$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right) + \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha-\beta}{2}\right)$$

Since above chord is focal chord,

$\therefore$ it passes through focus (ae, 0) or (– ae, 0)

$$\therefore \quad \pm e\cos\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\therefore \quad e = \left|\frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)}\right|$$ **Ans.**

Note : $\because$ $\pm e = \dfrac{\cos\frac{\alpha-\beta}{2}}{\cos\frac{\alpha+\beta}{2}}$

$$\pm e = \frac{1+\tan\frac{\alpha}{2}\,.\,\tan\frac{\beta}{2}}{1-\tan\frac{\alpha}{2}\,.\,\tan\frac{\beta}{2}}$$

Applying componendo and dividendo

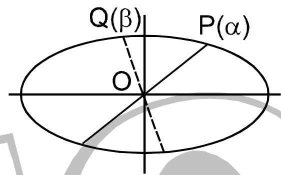
$$\frac{1\pm e}{\pm e-1} = \frac{2}{2\tan\frac{\alpha}{2}\,.\,\tan\frac{\beta}{2}}$$

$$\tan\frac{\alpha}{2}\ \tan\frac{\beta}{2} = \frac{1+e}{e-1} \text{ or } \frac{e-1}{1+e}$$

Solved Example # 11

Find the angle between two diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Whose extremities have eccentrici angle α and $\beta = \alpha + \frac{\pi}{2}$.

Solution.



Let ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Slope of OP = $m_1 = \frac{b\sin\alpha}{a\cos\alpha} = \frac{b}{a}\tan\alpha$

Slope of OQ = $m_2 = \frac{b\sin\beta}{a\cos\beta} = -\frac{b}{a}\cot\alpha$ given $\beta = \alpha + \frac{\pi}{2}$

$$\therefore \quad \tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right| = \left|\frac{\frac{b}{a}(\tan\alpha+\cot\alpha)}{1-\frac{b^2}{a^2}}\right| = \left|\frac{2ab}{(a^2-b^2)\sin 2\alpha}\right| \quad \textbf{Ans.}$$

Self Practice Problem

10. Find the sum of squares of two diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ whose extremitites have eccentric angles differ by $\frac{\pi}{2}$ and show that it is constant.

Ans. $4(a^2 + b^2)$

11. Show that the sum of squares of reciprocals of two perpendicular diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is constant. Find the constant also.

Ans. $\frac{1}{4}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)$

12. Find the locus of the foot of the perpendicular from the centre of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ on the chord joining two points whose eccentric angles differ by $\frac{\pi}{2}$.

Ans. $2(x^2 + y^2)^2 = a^2x^2 + b^2y^2$.

5. Position of a Point w.r.t. an Ellipse:

The point $P(x_1, y_1)$ lies outside, inside or on the ellipse according as ; $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$ > < or = 0.

Solved Example # 12

Check wether the point P(3, 2) lies inside or outside of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$.

Solution.

$S_1 \equiv \frac{9}{25} + \frac{4}{16} - 1 = \frac{9}{25} + \frac{1}{4} - 1 < 0$

$\therefore$ Point $P \equiv (3, 2)$ lies inside the ellipse. **Ans.**

Solved Example # 13

Find the set of value(s) of 'α' for which the point $P(\alpha, -\alpha)$ lies inside the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

Solution.

If $P(\alpha, -\alpha)$ lies inside the ellipse

$\therefore$ $S_1 < 0$

$\Rightarrow$ $\frac{\alpha^2}{16} + \frac{\alpha^2}{9} - 1 < 0$

$\Rightarrow$ $\frac{25}{144} \cdot \alpha^2 < 1$ $\Rightarrow$ $\alpha^2 < \frac{144}{25}$

Solved Example $\alpha \in \left(-\frac{12}{5}, \frac{12}{5}\right)$. **Ans.**

6. Line and an Ellipse:

The line $y = mx + c$ meets the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in two points real, coincident or imaginary according as c^2 is $<$ $=$ or $>$ $a^2m^2 + b^2$.

Hence $y = mx + c$ is tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if $c^2 = a^2m^2 + b^2$.

Solved Example # 14

Find the set of value(s) of 'λ' for which the line $3x - 4y + \lambda = 0$ intersect the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ at two distinct points.

Solution

Solving given line with ellipse, we get

$$\frac{(4y - \lambda)^2}{9 \times 16} + \frac{y^2}{9} = 1$$

$$\frac{2y^2}{9} - \frac{y\lambda}{18} + \frac{\lambda^2}{144} - 1 = 0$$

Since, line intersect the parabola at two distinct points,

$\therefore$ roots of above equation are real & distinct

$\therefore$ $D > 0$

$\Rightarrow$ $\frac{\lambda^2}{(18)^2} - \frac{8}{9} \cdot \left(\frac{\lambda^2}{144} - 1\right) > 0$

$\Rightarrow$ $-12\sqrt{2} < \lambda < 12\sqrt{2}$

Self Practice Problem

13. Find the value of 'λ' for which $2x - y + \lambda = 0$ touches the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$

Ans. $\lambda = \pm \sqrt{109}$

7. Tangents:

(a) Slope form: $y = mx \pm \sqrt{a^2m^2 + b^2}$ is tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ for all values of m.

(b) Point form : $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ is tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at (x_1, y_1).

(c) Parametric form: $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$ is tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $(a \cos \theta, b \sin \theta)$.

NOTE : (i) There are two tangents to the ellipse having the same m, i.e. there are two tangents parallel to any given direction. These tangents touches the ellipse at extremities of a diameter.

(ii) Point of intersection of the tangents at the point α & β is, $\left(a\,\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}},\ b\,\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$

(iii) The eccentric angles of the points of contact of two parallel tangents differ by π.

Solved Example # 15

Find the equations of the tangents to the ellipse $3x^2 + 4y^2 = 12$ which are perpendicular to the line $y + 2x = 4$.

Solution.

Slope of tangent = m = $\frac{1}{2}$

Given ellipse is $\frac{x^2}{4} + \frac{y^2}{3} = 1$

Equation of tangent whose slope is 'm' is

$y = mx \pm \sqrt{4m^2 + 3}$

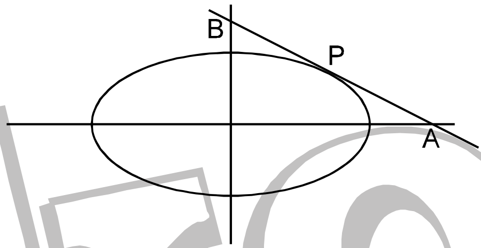
$\because$ $m = \frac{1}{2}$ $\therefore$ $y = \frac{1}{2} x \pm \sqrt{1+3}$

$2y = x \pm 4$

Solved Example # 16

A tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ touches at the point P on it in the first quadrant and meets the co-ordinate axes in A and B respectively. If P divides AB in the ratio 3 : 1, find the equation of the tangent.

Solution.



Let P ≡ (a cosθ, b sinθ)

$\therefore$ equation of tangent is

$\frac{x}{a} \cos\theta + \frac{y}{b} \sin\theta = 1$

A ≡ (a secθ, 0)

B ≡ (0, b cosecθ)

$\because$ P divide AB internally in the ratio 3 : 1

$\therefore$ $a \cos\theta = \frac{a \sec\theta}{4}$ $\Rightarrow$ $\cos^2\theta = \frac{1}{4}$ $\Rightarrow$ $\cos\theta = \frac{1}{2}$

and $b \sin\theta = \frac{3b \operatorname{cosec}\theta}{4}$ $\Rightarrow$ $\sin\theta = \frac{\sqrt{3}}{2}$

$\therefore$ tangent is $\frac{x}{2a} + \frac{\sqrt{3}y}{2b} = 1$ $\Rightarrow$ $bx + \sqrt{3}\, ay = 2ab$ **Ans.**

Solved Example # 17

Prove that the locus of the point of intersection of tangents to an ellipse at two points whose eccentric angle differ by a constant α is an ellipse.

Solution.

Let P (h, k) be the point of intersection of tangents at A(θ) and B(β) to the ellipse.

$$\therefore \quad h = \frac{a\cos\left(\frac{\theta+\beta}{2}\right)}{\cos\left(\frac{\theta-\beta}{2}\right)} \;\&\; k = \frac{b\sin\left(\frac{\theta+\beta}{2}\right)}{\cos\left(\frac{\theta-\beta}{2}\right)}$$

$$\Rightarrow \quad \left(\frac{h}{a}\right)^2 + \left(\frac{k}{b}\right)^2 = \sec^2\left(\frac{\theta-\beta}{2}\right)$$

but given that $\theta - \beta = \alpha$

$$\therefore \quad \text{locus is } \frac{x^2}{a^2\sec^2\left(\frac{\alpha}{2}\right)} + \frac{y^2}{b^2\sec^2\left(\frac{\alpha}{2}\right)} = 1$$ **Ans.**

Solved Example # 18

Find the locus of foot of perpendicular drawn from centre to any tangent to the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution.

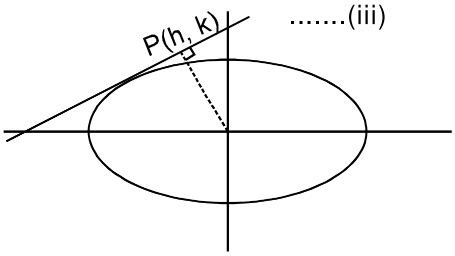
Let P(h, k) be the foot of perpendicular to a tangent $y = mx + \sqrt{a^2m^2 + b^2}$ ....(i)

from centre

$\therefore$ $\frac{k}{h} . m = -1$ $\Rightarrow$ $m = -\frac{h}{k}$ ....(ii)

$\because$ P(h, k) lies on tangent

$\therefore$ $k = mh + \sqrt{a^2m^2 + b^2}$ ....(iii)



from equation (ii) & (iii), we get

$$\left(k + \frac{h^2}{k}\right)^2 = \frac{a^2h^2}{k^2} + b^2$$

$\Rightarrow$ locus is $(x^2 + y^2)^2 = a^2x^2 + b^2y^2$ **Ans.**

Self Practice Problem

14. Show that the locus of the point of intersection of the tangents at the extremities of any focal chord of an ellipse is the directrix corresponding to the focus.

15. Show that the locus of the foot of the perpendicular on a varying tangent to an ellipse from either of its foci is a concentric circle.

16. Prove that the portion of the tangent to an ellipse intercepted between the ellipse and the directrix subtends a right angle at the corresponding focus.

17. Find the area of parallelogram formed by tangents at the extremities of latera recta of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Ans. $\frac{2a^3}{\sqrt{a^2 - b^2}}$

18. If y_1 is ordinate of a point P on the ellipse then show that the angle between its focal radius and tangent at it, is $\tan^{-1}\left(\frac{b^2}{aey_1}\right)$.

19. Find the eccentric angle of the point P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ tangent at which, is equally inclined to the axes.

Ans. $\theta = \pm \tan^{-1}\left(\frac{b}{a}\right), \pi - \tan^{-1}\left(\frac{b}{a}\right), -\pi + \tan^{-1}\left(\frac{b}{a}\right)$

8. Normals:

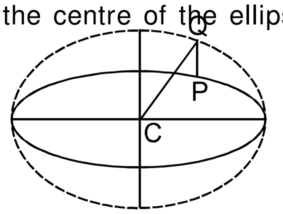
(i) Equation of the normal at (x_1, y_1) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$.

(ii) Equation of the normal at the point $(a\cos\theta, b\sin\theta)$ to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is; $ax.\sec\theta - by.\operatorname{cosec}\theta = (a^2 - b^2)$.

(iii) Equation of a normal in terms of its slope 'm' is $y = mx - \frac{(a^2 - b^2)m}{\sqrt{a^2 + b^2m^2}}$.

Solved Example # 19

P and Q are corresponding points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the auxiliary circles respectively. The normal at P to the ellipse meets CQ in R, where C is the centre of the ellipse. Prove that CR = a + b



Sol. Let $P \equiv (a\cos\theta, b\sin\theta)$

$\therefore$ $Q \equiv (a\cos\theta, a\sin\theta)$

Equation of normal at P is

$(a\sec\theta)\, x - (b\operatorname{cosec}\theta)\, y = a^2 - b^2$ ....(i)

equation of CQ is $y = \tan\theta \,.\, x$ ....(ii)

Solving equation (i) & (ii), we get

$(a - b)\, x = (a^2 - b^2)\cos\theta$

$x = (a + b)\cos\theta$, & $y = (a + b)\sin\theta$

$\therefore$ $R \equiv ((a + b)\cos\theta, (a + b)\sin\theta)$

$\therefore$ CR = a + b **Ans.**

Solved Example # 20

Find the shortest distance between the line x + y = 10 and the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$

Solution.

Shortest distance occurs between two non-intersecting curve always along common normal.
Let 'P' be a point on ellipse and Q is a point on given line for which PQ is common normal.

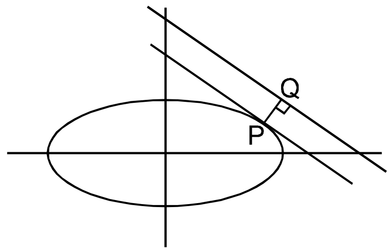
$\therefore$ Tangent at 'P' is parallel to given line

$\therefore$ Equation of tangent parallel to given line is ($y = mx \pm \sqrt{a^2m^2 + b^2}$)

$y = -x \pm 5$

$\Rightarrow$ $x + y + 5 = 0$ or $x + y - 5 = 0$

$\therefore$ minimum distance = distance between $x + y - 10 = 0$ & $x + y - 5 = 0$



$\Rightarrow$ shortest distance $= \dfrac{|10-5|}{\sqrt{1+1}}$

$= \dfrac{5}{\sqrt{2}}$ **Ans.**

Solved Example # 21

Prove that, in an ellipse, the distance between the centre and any normal does not exceed the difference between the semi-axes of the ellipse.

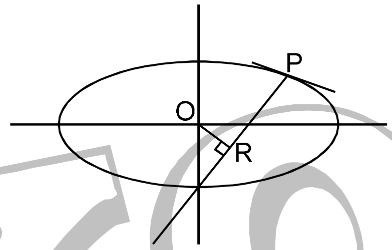
Solution.

Let the equation of ellipse is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

Equation of normal at P (θ) is

$(a\sec\theta)x - (b\operatorname{cosec}\theta)y - a^2 + b^2 = 0$

distance of normal from centre



$$= OR = \frac{|a^2 - b^2|}{\sqrt{a^2 + b^2 + (a\tan\theta)^2 + (b\cot\theta)^2}}$$

$$= \frac{|a^2 - b^2|}{\sqrt{(a+b)^2 + (a\tan\theta - b\cot\theta)^2}}$$

$\because$ $(a + b)^2 + (a\tan\theta - b\cot\theta)^2 \ge (a + b)^2$ or $\le \dfrac{|a^2 - b^2|}{\sqrt{(a+b)^2}}$

$|OR| \le (a - b)$ **Ans.**

Self Practice Problem

20. Find the value(s) of 'k' for which the line $x + y = k$ is a normal to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

Ans. $k = \pm\sqrt{\dfrac{(a^2 - b^2)^2}{a^2 + b^2}}$

21. If the normal at the point P(θ) to the ellipse $\dfrac{x^2}{14} + \dfrac{y^2}{5} = 1$ intersects it again at the point Q(2θ) then $\cos\theta =$

(A*) $-\dfrac{2}{3}$ (B) $\dfrac{2}{3}$ (C) $-\dfrac{6}{7}$ (D) $\dfrac{6}{7}$

9. Pair of Tangents:

The equation to the pair of tangents which can be drawn from any point (x_1, y_1) to the ellipse

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is given by: $SS_1 = T^2$ where :

$S \equiv \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1$; $S_1 = \dfrac{x_1^2}{a^2} + \dfrac{y_1^2}{b^2} - 1$; $T \equiv \dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} - 1$.

Solved Example # 22

How many real tangents can be drawn from the point (4, 3) to the ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$. Find the equation these tangents & angle between them.

Solution.

Given point $P \equiv (4, 3)$

ellipse $S \equiv \dfrac{x^2}{16} + \dfrac{y^2}{9} - 1 = 0$

$\because \quad S_1 \equiv \dfrac{16}{16} + \dfrac{9}{9} - 1 = 1 > 0$

$\Rightarrow$ Point $P \equiv (4, 3)$ lies outside the ellipse.

$\therefore$ Two tangents can be drawn from the point $P(4, 3)$.

Equation of pair of tangents is

$SS_1 = T^2$

$$\Rightarrow \left(\frac{x^2}{16}+\frac{y^2}{9}-1\right) . 1 = \left(\frac{4x}{16}+\frac{3y}{9}-1\right)^2$$

$$\Rightarrow \frac{x^2}{16} + \frac{y^2}{9} - 1 = \frac{x^2}{16} + \frac{y^2}{9} + 1 + \frac{xy}{6} - \frac{x}{2} - \frac{2y}{3} \qquad \Rightarrow \qquad - xy + 3x + 4y - 12 = 0$$

$\Rightarrow \quad (4 - x)(y - 3) = 0 \qquad \Rightarrow \qquad x = 4 \ \& \ y = 3$

and angle between them = $\dfrac{\pi}{2}$ **Ans.**

Sol. Ex. # 23: Find the locus of point of intersection of perpendicular tangents to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2} = 1$

Solution. Let $P(h, k)$ be the point of intersection of two perpendicular tangents

$$\Rightarrow \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}-1\right) = \left(\frac{hx}{a^2}+\frac{ky}{b^2}-1\right)^2$$

$$\Rightarrow \frac{x^2}{a^2}\left(\frac{k^2}{b^2}-1\right) + \frac{y^2}{a^2}\left(\frac{h^2}{a^2}-1\right) + \text{........} = 0 \qquad \text{.........(i)}$$

Since equation (i) represents two perpendicular lines

$$\therefore \quad \frac{1}{a^2}\left(\frac{k^2}{b^2}-1\right) + \frac{1}{b^2}\left(\frac{h^2}{a^2}-1\right) = 0$$

$\Rightarrow \quad k^2 - b^2 + h^2 - a^2 = 0 \qquad \Rightarrow \qquad$ locus is $x^2 + y^2 = a^2 + b^2$ **Ans.**

Self Practice Problem :

22. Find the locus of point of intersection of the tangents drawn at the extremities of a focal chord of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. **Ans.** $x = \pm \dfrac{a}{e}$

10. Director Circle:

Locus of the point of intersection of the tangents which meet at right angles is called the **Director Circle.** The equation to this locus is $x^2 + y^2 = a^2 + b^2$ i.e. a circle whose centre is the centre of the ellipse & whose radius is the length of the line joining the ends of the major & minor axes.

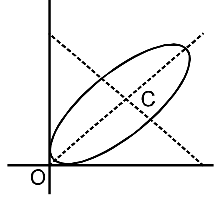
Solved Example # 24

An ellipse slides between two perpendicular lines. Show that the locus of its centre is a circle.

Solution : Let length of semi-major and semi-minor axis are 'a' and 'b' and centre is $C \equiv (h, k)$

Since ellipse slides between two perpendicular lines, there for point of intersection of two perpendicular tangents lies on director circle.

Let us consider two perpendicular lines as x & y axes



$\therefore$ point of intersection is origin $O \equiv (0, 0)$

$\therefore$ OC = radius of director circle

$\therefore \quad \sqrt{h^2+k^2} = \sqrt{a^2+b^2}$

$\Rightarrow$ locus of $C \equiv (h, k)$ is

$\Rightarrow \quad x^2 + y^2 = a^2 + b^2$ which is a circle

Self Practice Problem

A tangent to the ellipse $x^2 + 4y^2 = 4$ meets the ellipse $x^2 + 2y^2 = 6$ at P and Q. Prove that the tangents at P and Q of the ellipse $x^2 + 2y^2 = 6$ are at right angles.

11. Chord of Contact:

Equation to the chord of contact of tangents drawn from a point $P(x_1, y_1)$ to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is

$T = 0$, where $\quad T = \dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} - 1$

Solved Example # 25

If tangents to the parabola $y^2 = 4ax$ intersect the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ at A and B, then find the locus of point of intersection of tangents at A and B.

Solution: Let P ≡ (h, k) be the point of intersection of tangents at A & B

$\therefore$ equation of chord of contact AB is $\dfrac{xh}{a^2} + \dfrac{yk}{b^2} = 1$ ....(i)

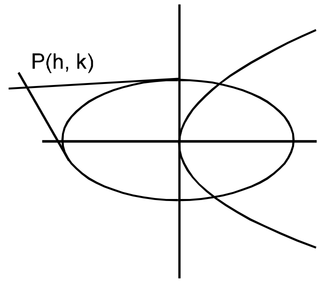
which touches the parabola equation of tangent to parabola $y^2 = 4ax$

$$y = mx + \frac{a}{m}$$

$\Rightarrow$ $mx - y = -\dfrac{a}{m}$ ....(ii)

equation (i) & (ii) as must be same

$$\therefore \quad \frac{m}{\left(\frac{h}{a^2}\right)} = \frac{-1}{\left(\frac{k}{b^2}\right)} = \frac{-\frac{a}{m}}{1}$$



$\Rightarrow$ $m = -\dfrac{h}{k}\,\dfrac{b^2}{a^2}$ & $m = \dfrac{ak}{b^2}$

$\therefore$ $-\dfrac{hb^2}{ka^2} = \dfrac{ak}{b^2}$

$\Rightarrow$ locus of P is $y^2 = -\dfrac{b^4}{a^3}\,.\,x$ **Ans.**

Self Practice Problem

23. Find the locus of point of intersection of tangents at the extremities of normal chords of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. **Ans.** $\dfrac{a^6}{x^2} + \dfrac{b^6}{y^2} = (a^2 - b^2)^2$

24. Find the locus of point of intersection of tangents at the extremities of the chords of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ subtending a right angle at its centre.

Ans. $\dfrac{x^2}{a^4} + \dfrac{y^2}{b^4} = \dfrac{1}{a^2} + \dfrac{1}{b^2}$

12. Chord with a given middle point:

Equation of the chord of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ whose middle point is (x_1, y_1) is $T = S_1$,

where $S_1 = \dfrac{x_1^2}{a^2} + \dfrac{y_1^2}{b^2} - 1$; $T \equiv \dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} - 1$.

Solved Example # 26

Find the locus of the mid - point of focal chords of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

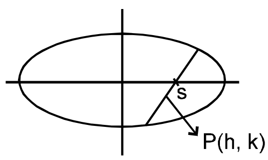
Solution: Let P ≡ (h, k) be the mid-point

$\therefore$ equation of chord whose mid-point is given $\dfrac{xh}{a^2} + \dfrac{yk}{b^2} - 1 = \dfrac{h^2}{a^2} + \dfrac{k^2}{b^2} - 1$

since it is a focal chord,

$\therefore$ it passes through focus, either (ae, 0) or (–ae, 0)

If it passes trhrough (ae, 0)



$\therefore$ locus is $\dfrac{ex}{a} = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$

If it passes through (–ae, 0)

$\therefore$ locus is $-\dfrac{ex}{a} = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$ **Ans.**

Solved Example # 27: Find the condition on 'a' and 'b' for which two distinct chords of the ellipse $\dfrac{x^2}{2a^2} + \dfrac{y^2}{2b^2} = 1$ passing through (a, – b) are bisected by the line x + y = b.

Solution: Let the line x + y = b bisect the chord at $P(\alpha, b - \alpha)$

$\therefore$ equation of chord whose mid-point is $P(\alpha, b - \alpha)$

$$\frac{x\alpha}{2a^2} + \frac{y(b-\alpha)}{2b^2} = \frac{\alpha^2}{2a^2} + \frac{(b-\alpha)^2}{2b^2}$$

Since it passes through (a, –b)

$\therefore \quad \dfrac{\alpha}{2a} - \dfrac{(b-\alpha)}{2b} = \dfrac{\alpha^2}{2a^2} + \dfrac{(b-\alpha)^2}{2b^2}$

$\Rightarrow \quad \left(\dfrac{1}{a}+\dfrac{1}{b}\right)\alpha - 1 = \alpha^2\left(\dfrac{1}{a^2}+\dfrac{1}{b^2}\right) - \dfrac{2}{b}\alpha + 1 \qquad \Rightarrow \qquad \alpha^2\left(\dfrac{1}{a^2}+\dfrac{1}{b^2}\right) - \left(\dfrac{3}{b}+\dfrac{1}{a}\right)\alpha + 2 = 0$

since line bisect two chord

$\therefore$ above quadratic equation in α must have two distinct real roots

$\therefore \quad \left(\dfrac{3}{b}+\dfrac{1}{a}\right)^2 - 4\left(\dfrac{1}{a^2}+\dfrac{1}{b^2}\right).\,2 > 0$

$\Rightarrow \quad \dfrac{9}{b^2} + \dfrac{1}{a^2} + \dfrac{6}{ab} - \dfrac{8}{a^2} - \dfrac{8}{b^2} > 0 \qquad \Rightarrow \qquad \dfrac{1}{b^2} - \dfrac{7}{a^2} + \dfrac{6}{ab} > 0$

$\Rightarrow \quad a^2 - 7b^2 + 6ab > 0$

$\Rightarrow \quad a^2 > 7b^2 - 6ab$ which is the required condition.

Self Practice Problem

25. Find the equation of the chord $\dfrac{x^2}{36} + \dfrac{y^2}{9} = 1$ which is bisected at (2, 1).

Ans. $x + 2y = 4$

26. Find the locus of the mid-points of normal chords of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Ans. $\left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}\right)^2\left(\dfrac{a^6}{x^2}+\dfrac{b^6}{y^2}\right) = (a^2 - b^2)^2$

27. Find the length of the chord of the ellipse $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$ whose middle point is $\left(\dfrac{1}{2}, \dfrac{2}{5}\right)$

Ans. $\dfrac{7}{5}\sqrt{41}$

13. Important High Lights :

Refering to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

- If P be any point on the ellipse with S & S′ as its foci then ℓ(SP) + ℓ(S′P) = 2a.
- The tangent & normal at a point P on the ellipse bisect the external & internal angles between the focal distances of P. This refers to the well known reflection property of the ellipse which states that rays from one focus are reflected through other focus & vice–versa. Hence we can deduce that the straight lines joining each focus to the foot of the perpendicular from the other focus upon the tangent at any point P meet on the normal PG and bisects it where G is the point where normal at P meets the major axis.
- The product of the length's of the perpendicular segments from the foci on any tangent to the ellipse is b^2 and the feet of these perpendiculars lie on its auxiliary circle and the tangents at these feet to the auxiliary circle meet on the ordinate of P and that the locus of their point of intersection is a similiar ellipse as that of the original one.
- The portion of the tangent to an ellipse between the point of contact & the directrix subtends a right angle at the corresponding focus.
- If the normal at any point P on the ellipse with centre C meet the major & minor axes in G & g respectively, & if CF be perpendicular upon this normal, then

  (i) PF. PG = b^2 (ii) PF. Pg = a^2

  (iii) PG. Pg = SP. S′P (iv) CG. CT = CS^2

  (v) locus of the mid point of Gg is another ellipse having the same eccentricity as that of the original ellipse.

  [where S and S′ are the focii of the ellipse and T is the point where tangent at P meet the major axis]
- The circle on any focal distance as diameter touches the auxiliary circle. Perpendiculars from the centre upon all chords which join the ends of any perpendicular diameters of the ellipse are of constant length.
- If the tangent at the point P of a standard ellipse meets the axis in T and t and CY is the perpendicular on it from the centre then,

  (i) T t. PY = $a^2 - b^2$ and

  (ii) least value of T t is a + b.

